Int J Theor Phys (2008) 47: 911–927
DOI 10.1007/s10773-007-9517-9

Time Evolution of Spontaneous Emission Rate of Two Undistinguished Radiators

Nicolae Enaki · Elena Galeamov

Received: 12 January 2007 / Accepted: 4 August 2007 / Published online: 14 September 2007

Abstract The behavior of the system of radiators at short and long time intervals in comparison with the retardation between them is studied. The entanglement behavior of atomic states in the process of spontaneous emission is determined. It is demonstrated that at a short time interval the rate of spontaneous emission in an oscillatory manner tends to the exponential law of spontaneous emission. The simple kinetic equation, which describes this stage of system evolution, is obtained.

1 Introduction

The processes of collective spontaneous decay of time-dependent quantum systems and the formation of two photons as wave packets are of great interest to quantum optics. The latest studies in these fields result in a deeper understanding of the main principles of quantum mechanics of simultaneous emission of two indistinguishable single photons by different emitters [4, 16].

The first report on non-exponential spontaneous decay of quasi-stationary states can be found in the work by Khalfin [11–13], who demonstrated, that the corrections to the exponential law of spontaneous decay behave like $1/t$, with t the time, basing his reasoning on the fact that the energy distribution density is semi-finite. Simultaneously with the development of model atom-field interaction Hamiltonians more exact expressions for the corrections to the exponential spontaneous decay law appeared in literature [5, 15]. The papers by Wilkievich and Eberly [21], and Atkins and Wooloy [2], should also be mentioned, since the researchers point out to the need to allow for the time lag within the atom when the corrections to the exponential law are taken into account. The calculations show that in the time intervals that are large compared to the reciprocal photon emission frequency ω^{-1} the correction to the exponential spontaneous decay law behaves like $1/t$ and constitutes a small addition to the exponential. The effect of the anti-resonant terms in the Hamiltonian on the

---

N. Enaki (✉) · E. Galeamov
Institute of Applied Physics, Academy of Sciences of Moldova, Academiei str. 5, Chisinau 2028, Republic of Moldova
e-mail: quantumopticslab@gmail.com

Lamb shift of the energy levels in the process of spontaneous decay is also of interest. In the papers Seke and Herfort [19, 20], Facchi and Pascazio [8] an interesting mathematical study of this problem is given.

In the paper [7] the equation for the mean value of the population difference has been obtained. This equation implies that the emerging wave packets (photons) in the time interval $t > 1/\omega$ have a strong effect not only on the excited state of the radiators (atoms, nuclei) but also on the ground state. The coefficient of the stimulating action of the emerging photon field is of the same order of magnitude as the spontaneous transition coefficient. At early time periods, the emerging wave packet interacts with the excited and ground states. For times t of order to the reciprocal photon emission frequency the probability of stimulated action on the ground and excited states of the atom is of the same order of magnitude as the reciprocal spontaneous decay time of the atoms. Only for short time periods the probability tends to small corrections to the traditional law of spontaneous decay, which are well known in literature [7]. The cooperative behavior of two and more atoms in interaction with electromagnetic field (EMF) vacuum remains in the attention of recent studies [3, 6, 9, 23]. The problem of non-locality of electromagnetic field plays an important role in the process of two atoms emission. It's interesting to regard this problem following the method of non-locality proposed in paper [7].

In this paper we propose a theory that describes the spontaneous decay of two hydrogen-like atoms on time scales that are small or large compared to the inverse value of retardation time between the radiators. On short-time scales the spontaneous decay rate is found to tend in an oscillatory manner to the exponential law of spontaneous decay. The process of energy exchange in time and space between two radiators placed at distance D is discussed. The behavior of non-locality of EMF in the process of spontaneous emission is observed. This effect gives us non-causality terms in the exchange integrals between these radiators. In order to take into account all specific properties of quantum non-locality of two radiators located on distance D, we propose the technique of elimination of EMF operators without dipole approximation of these radiators in interaction with the field. The system of equations, which describes the interaction of radiators with EMF vacuum, is obtained. The behavior of this system of radiators at short and long time intervals in comparison with D/c is carefully studied. The corrections to spontaneous emission times and cooperative emission times are studied. The specific problem appears at the moment when retardation effects occur when time achieves value D/c. In this case the classical and non-classical effects work together in the acceleration rate of the collective spontaneous emission.

It should be noted, that during the collapse of two excited atoms into the ground state the vacuum fluctuations of the EMF are of the order of the emitted photons wavelength. The length of the photons wave packet considerably exceeds the quantum size of the atoms proper (we mean the first Bohr orbit) and may be comparable with the distance between the radiators. Hence, owing to the finiteness of the group velocity the wave has not time enough to leave the atoms instantaneously. In small time intervals, there is an exchange of energy between an excited non-stationary state of two atoms and a state of the electromagnetic field. This effect generates another effect connected with exchange of energies between the radiators.

In order to allow for the time lag within the atoms a new method is suggested, its idea is based on integrating the right-hand sides of the kinetic equations over the frequencies of the EMF and the coordinates of two radiators. The system of equations, obtained in non-dipole approximation, takes into account the anti-resonant terms in interaction Hamiltonian. The time behavior of interaction spontaneous emission rate, which describes the interaction of individual atoms with EMF vacuum and cooperative rate, is analytically obtained. The time evolution of spontaneous emission is given.

2 Master Equation of Two Undistinguished Atoms in Interaction with Electromagnetic Field Vacuum

The process of studying the interaction of two hydrogen-like atoms and a variable EMF is simplified by shifting the origin to the center of the electron and nucleus. We can describe the motion of the electron in the central field of the nucleus and the variable EMF by Hamiltonian

$$H = H_0 + H_I, \tag{1a}$$

$$H_0 = \sum_{i=1,2}\left(\frac{p_i^2}{2\mu} - \frac{ze^2}{r_i}\right) + \frac{1}{2}\int dV(\mathbf{B}^2 + \mathbf{E}^2), \tag{1b}$$

$$H_I = -\sum_{j=1,2}\left[\frac{e}{c\mu}(\mathbf{A}(\mathbf{r}_j, t), \mathbf{p}_j) + \frac{e^2}{2\mu c^2}\mathbf{A}^2(\mathbf{r}_j, t)\right]. \tag{1c}$$

Here $\mathbf{r}$ and $\mathbf{p} = -i\hbar\nabla$ are the electron position and momentum vectors with respect to the center of the mass, and ze is the charge of the nuclei, $\mathbf{E}$ and $\mathbf{B}$ are the vectors of the electric and magnetic field.

The aim of this paper is to allow the retardation of the emitted EMF in the process of collapse of the excited electron to the ground state. Hence, in the interaction Hamiltonian (1c) in the non-relativistic approximation we ignore the term proportional to the square of the vector potential, which is common in practice. The positions $\mathbf{r}_A$ and $\mathbf{r}_B$ of the electrons of A and B of two hydrogen-like atoms in the space situated at the distance D are represented in Fig. 1. In order to simplify the Hamiltonian (1), let's consider $\mathbf{r}_1$ and $\mathbf{r}_2$ the positions of the parent electrons for atoms A and B relative to the nucleus.

As the problem of corrections to the spontaneous emission was studied only for one atom, in the literature appears a problem connected with two atoms in which quantification procedure *affects the classical causality* in the process of energy exchange between the atoms through the EMF vacuum. Let us demonstrate this below. The Heisenberg equation for atomic operators which describes the spontaneous emission, must be averaged taking into account the initial atomic and field states.

Indeed, in according with retardation solution of Maxwell equation

$$\left(\triangle - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A}(t, \mathbf{r}) = \frac{4\pi}{c}\mathbf{J}^{\perp}, \tag{2a}$$

where

$$\mathbf{J}^{\perp}(t, \mathbf{r}) = \frac{e}{\mu}\sum_{j=A,B}\mathbf{p}_{\perp}(t, \mathbf{r})\delta^3(\mathbf{r} - \mathbf{r}_j),$$

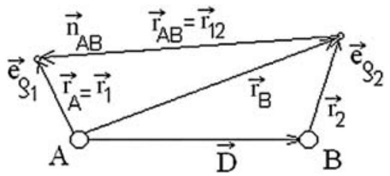


Fig. 1 The positions r_A and r_B of the electrons of A and B of two hydrogen-like atoms in the space situated at the distance D

the vector potential of electromagnetic field satisfies the causality principle

$$\mathbf{A}(t,\mathbf{r}) = \frac{1}{c}\int \frac{[\mathbf{J}^{\perp}(t,\mathbf{r})]_{\text{ret}}}{|\mathbf{r}-\mathbf{r}'|} d^3r' + \mathbf{A}_{\text{free}}. \tag{2b}$$

Here $\mathbf{A}_{\text{free}}$ is the free solution of (2). Taking into account the quantification of EMF in interaction with the atoms

$$H_f = \sum_k \hbar\omega_k\{a_k^+(t)a_k(t) + 1/2\} - \frac{e}{c\mu}\sum_{j=A,B}(\mathbf{p}_j, \mathbf{g}_k)\{a_k^+(t)\exp[-i(\mathbf{k},\mathbf{r})] + h.c.\} \tag{3}$$

one can represent the Heisenberg solutions of positive and negative frequency parts operators through the EMF vacuum and source contribution respectively, $\mathbf{A}(t,\mathbf{r}) = \mathbf{A}^{(+)}(t,\vec{r}) + \mathbf{A}^{(-)}(t,\vec{r})$, where

$$\mathbf{A}^{(+)}(t,\mathbf{r}) = \mathbf{A}^{(+)}_{\text{free}}(t,\vec{r}) + \mathbf{A}^{(+)}_{s}(t,\mathbf{r}), \qquad \mathbf{A}^{(-)}(t,\mathbf{r}) = [\mathbf{A}^{(+)}(t,\mathbf{r})]^+. \tag{4}$$

Here

$$\mathbf{A}^{(+)}_{\text{free}}(t,\mathbf{r}) = \sum_k \mathbf{g}_k a_k(0)\exp[-i\omega_k t + i(\mathbf{k},\mathbf{r})],$$

$$\mathbf{A}^{(+)}_{s}(t,\mathbf{r}) = \frac{i}{\hbar}\frac{e}{c\mu}\sum_k\sum_{j=A,B}\int_0^t dt'\exp[i\omega_k(t'-t) + i(\mathbf{k},\mathbf{r}-\mathbf{r}_j)](\mathbf{p}_j(t'),\mathbf{g}_k)\mathbf{g}_k.$$

The operators of EMF satisfy the commutation relations

$$[a_k, a_{k'}^+] = \delta_{kk'}, [a_k, a_{k'}] = [a_k^+, a_{k'}^+] = 0,$$

$\mathbf{g}_k = \sqrt{2\pi c^2\hbar/V\omega_k}\mathbf{e}_\lambda$, V is the quantization volume, $\mathbf{e}_\lambda$ is the photon polarization vector ($\lambda = 1, 2$), $k = (\mathbf{k}, \lambda)$. We observe that the positive (or negative) part of the of new solution (4) does not satisfy the wave equations (2a).

The problem with causality between the radiators in the kinetic equation appears in the process of elimination of free parts of solution (4). If we assume that at the initial time moment, $t = 0$, the interaction between the hydrogen-like atoms and the electromagnetic field vacuum is switched on, and the electrons are prepared in the quantum state $|n = 2, l = 1, m = 0\rangle$ (here n, l and m are, respectively, the principal, orbital, and magnetic quantum numbers), than the collective spontaneous emission can be obtained from the Hamiltonian

$$\begin{aligned} H = &\sum_{j=A,B}\hbar\omega_0 R_{zj} + \sum_k \hbar\omega_k(a_k^+a_k + 1/2) \\ &+ \sum_{j=A,B}\sum_k g_k\{(\varphi^j_{12}(k)a_k^+ + \varphi^j_{12}(-k)a_k)R_j^- + (\varphi^j_{21}(k)a_k + \varphi^j_{21}(-k)a_k^+)R_j^+\}. \end{aligned} \tag{5}$$

Here the interaction constant of atoms with the electromagnetic field is represented through matrix elements

$$\varphi^A_{12}(k) = \frac{ie\hbar}{c\mu}\int dV_1\psi_{100}(r_1)(\mathbf{e}_\lambda, \nabla_1)\exp[-i(\mathbf{k},\mathbf{r}_A)]\psi_{210}(r_1),$$

$$\varphi^A_{21}(k) = \frac{ie\hbar}{c\mu}\int dV_2\psi_{210}(r_2)(\mathbf{e}_\lambda, \nabla_2)\exp[i(\mathbf{k},\mathbf{r}_B)]\psi_{100}(r_2),$$

R^+, R^-, and R_z, are the operators of the atomic subsystem satisfying the commutation relations

$$[R_i^+, R_j^-] = 2R_{zj}\delta_{ij}, \qquad [R_i^\pm, R_{zj}] = \mp R_i^\pm \delta_{ij},$$

$\psi_{100}(r) = \exp(-\rho)/\sqrt{\pi}$ and $\psi_{210}(r) = \rho\cos\theta\exp(-\rho/2)/\ (4\sqrt{2\pi})$ are the wave function of the ground ($|1s\rangle$) and the excited ($|2p\rangle$) states of two hydrogen-like atoms, $\rho = |r|/a$, $a = \hbar/(\mu e^2 z)$ is the first Bohr radius, and $\hbar\omega_0$ is the energy separation of the ground $|1s\rangle$ and excited $|2p\rangle$ states. In deriving the Hamiltonian (4) from (1) we allowed for the anti-resonant terms in the interaction with the electromagnetic field. It should be noted that to ensure the convergence of integrals when calculating the corrections to the spontaneous decay we did not employ the common dipole approximation $\varphi_{12} \simeq i\omega_0(\mathbf{e}_\lambda, \mathbf{d})\hbar/c$ proposed in literature [1, 14] (**d** is the dipole moment of the transition). The properties of these matrix elements are: $[\varphi_{12}^j(k)]^* = \varphi_{21}^j(k)$.

Let us consider an operator of the atomic subsystem $O(t)$, that depends on the atomic operators. The Heisenberg equation for the mean value of atomic operator, $O(t)$, can be written

$$\frac{d\langle O(t)\rangle}{dt} = \sum_{j=A,B}\Bigg[i\omega_0\langle[R_{zj}(t), O(t)]\rangle + \frac{i}{\hbar}\sum_k g_k[\langle a_k^+(t)[O(t), R_j^-(t)\varphi_{12}^j(k) + R_j^+(t)\varphi_{21}^j(-k)]\rangle - \langle[R_j^-(t)\varphi_{12}^j(-k) + \varphi_{21}^j(k)R_j^+(t), O(t)]a_k(t)\rangle]\Bigg]. \tag{6}$$

Here the averages $\langle\ldots\rangle$ take place at the initial state of the system $\psi(t=0) = |0\rangle|A\rangle$, where $|A\rangle$ and $|0\rangle$ are atomic and field initial states respectively.

The procedure of elimination of the free positive, $A_{\text{free}}^{(+)}(t, r)$, and negative, $A_{\text{free}}^{(-)}(t, r)$, frequency parts from expressions (4) connected with annihilation and creation operator destroy the classical causality. Taking into consideration that the action of this operators on the vacuum state of EMF, $\mathbf{A}_{\text{free}}^{(+)}(t, \mathbf{r})|0\rangle$ and $\langle 0|A_{\text{free}}^{(+)}(t, \mathbf{r}) = 0$, one can observe, that the mean value of the operator $\mathbf{A}(t, \mathbf{r})\hat{O}(t)$ is represented through the source parts in the following way

$$\langle 0|\mathbf{A}(t, \mathbf{r})\hat{O}(t)|0\rangle = \langle 0|\mathbf{A}^{(-)(s)}(t, \mathbf{r})\hat{O}(t)|0\rangle + \langle 0|\hat{O}(t)\mathbf{A}^{(+)(s)}(t, \mathbf{r})|0\rangle.$$

Each source part (positive or negative) are situated in the different part of correlators and does not satisfy the wave equations (2a) (as a result the retardation principle becomes violated). Of course, doing one commutation of source part of the operator with the atomic operator, one can restore the position of negative and positive source parts, but in this case remain the none-zero quantum commutation, which is responsible for non-local behavior of the system. After substitution of the solutions of Heisenberg equation for the electromagnetic field operators $a_k^+(t)$ and $a_k(t)$

$$a_k^+(t) = a_k^+(0)\exp(i\omega_k t) + \sum_{j=A,B}\frac{ig_k}{\hbar}\int_0^t d\tau[\varphi_{12}^j(-k)R_j^-(t-\tau) + \varphi_{21}^j(k)R_j^+(t-\tau))]\exp(i\omega_k\tau), \quad a_k(t) = [a_k^+(t)]^+,$$

into (6), we can eliminate the free parts of these operators. Accordingly with this, one can obtain the following master equation for atomic subsystem operator $O(t)$

$$\frac{d\langle O(t)\rangle}{dt} = \sum_{j=A,B} i\omega_0\langle[R_{zj}(t), O(t)]\rangle + \sum_k\sum_{i=A,B}\sum_{j=A,B}\frac{g_k^2}{\hbar^2}\int_0^t d\tau$$

$$
\begin{aligned}
&\times [\{\langle(\varphi^i_{21}(k)R^+_i(t-\tau)+\varphi^i_{12}(-k)R^-_i(t-\tau))\\
&\times [O(t), R^-_j(t)\varphi^j_{12}(k)+R^+_j(t)\varphi^j_{21}(-k)]\rangle\}\exp(i\omega_k\tau)\\
&+\{\langle[R^-_j(t)\varphi^j_{12}(-k)+\varphi^j_{21}(k)R^+_j(t), O(t)]\\
&\times(\varphi^i_{21}(-k)R^+_i(t-\tau)+\varphi^i_{12}(k)R^-_i(t-\tau))\rangle\}\exp(-i\omega_k\tau)]. \qquad (7)
\end{aligned}
$$

The right-hand sides of (7) contain the expression

$$J(\tau)=\sum_k \varphi^j_{21}(k)\varphi^j_{12}(k)\exp(i\omega_k\tau)\frac{g_k^2}{\hbar^2} \qquad (8)$$

which describes the interaction of j atom with itself through electromagnetic field vacuum. Examining the undistinguished radiators, we consider that $J(\tau)=J_{AA}(\tau)=J_{BB}(\tau)$. The exchange integral

$$J_{AB}(\tau)=\sum_k \varphi^A_{21}(k)\varphi^B_{12}(k)\exp(i\omega_k\tau)\frac{g_k^2}{\hbar^2} \qquad (9)$$

describes the cooperative interaction between A and B atoms. Both expressions (8) and (9) are integrated over neither the spatial variables of the atom nor $\mathbf{k}$. The exchange integral (8) was calculated in paper [7]. Since the emerging wave packet has the size of approximately the radiation wavelength and is larger than the first Bohr orbit by a factor of several thousands, exact integration of (7) would make it possible to study the behavior of the spontaneous decay rate at the initial stage of emission. As the sizes of the atoms are about Bohr radius, the photon cannot leave the atom immediately due to the finiteness of the photon group velocity. Directing the polar axis along the vector $\mathbf{r}_A-\mathbf{r}_B$, the sum over the polarization λ and the integral over all the directions of $\mathbf{k}$ in (9) can be calculated

$$
\begin{aligned}
&\sum_{\lambda=1}^{2}\int d\Omega_k \exp[i(\mathbf{k},\mathbf{r}_A-\mathbf{r}_B)](\mathbf{e}_\lambda,\mathbf{e}_{\rho_1})(\mathbf{e}_\lambda,\mathbf{e}_{\rho_2})\\
&\quad=2\pi\left[\frac{\sin kr_{AB}}{kr_{AB}}[(\mathbf{e}_{\rho_1},\mathbf{e}_{\rho 2})+(\mathbf{n}_{AB},\mathbf{e}_{\rho_1})(\mathbf{n}_{AB},\mathbf{e}_{\rho_2})]\right.\\
&\qquad\left.-\frac{1}{r_{AB}^2}\frac{\partial^2}{\partial k^2}\frac{\sin kr_{AB}}{kr_{AB}}[(\mathbf{e}_{\rho_1},\mathbf{e}_{\rho 2})-3(\mathbf{n}_{AB},\mathbf{e}_{\rho_1})(\mathbf{n}_{AB},\mathbf{e}_{\rho_2})]\right], \qquad (10)
\end{aligned}
$$

where $\mathbf{e}_{\rho_1}=\mathbf{r}_1/r_1$, $\mathbf{e}_{\rho_2}=\mathbf{r}_2/r_2$, $\mathbf{n}_{AB}=\frac{\mathbf{r}_A-\mathbf{r}_B}{r_{AB}}$, $r_{AB}=\sqrt{(\mathbf{r}_A-\mathbf{r}_B)^2}$. The same expression can be obtained in the right-hand side of integral (8)

$$
\begin{aligned}
&\sum_{\lambda=1}^{2}\int d\Omega_k \exp[i(\mathbf{k},\mathbf{r})](\mathbf{e}_\lambda,\mathbf{e}_{\rho_1})(\mathbf{e}_\lambda,\mathbf{e}_{\rho_1'})\\
&\quad=2\pi\left[\frac{\sin k|\mathbf{r}|}{k|\mathbf{r}|}[(\mathbf{e}_{\rho_1},\mathbf{e}_{\rho_1'})+(\mathbf{n},\mathbf{e}_{\rho_1})(\mathbf{n},\mathbf{e}_{\rho_1'})]\right.\\
&\qquad\left.-\frac{1}{|\mathbf{r}|^2}\frac{\partial^2}{\partial k^2}\frac{\sin k|\mathbf{r}|}{k|\mathbf{r}|}[(\mathbf{e}_{\rho_1},\mathbf{e}_{\rho_1'})-3(\mathbf{n},\mathbf{e}_{\rho_1})(\mathbf{n},\mathbf{e}_{\rho_1'})]\right], \qquad (11)
\end{aligned}
$$

where $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_1'$, $\mathbf{n}=\mathbf{r}/r$, $\mathbf{e}_{\rho_1}=\mathbf{r}_1/r_1$, $\mathbf{e}_{\rho_1'}=\mathbf{r}_1'/r_1'$.

From Fig. 1 it is observed that $\mathbf{r}_B = \mathbf{r}_2 + \mathbf{D}$ and $\mathbf{r}_{AB} = \mathbf{r}_A - \mathbf{r}_B$. Taking into consideration that the distance $\mathbf{D}$ between the atoms is larger than the first Bohr radius ($|\mathbf{r}_2|, |\mathbf{r}_1| \sim a$), we can expand (11) in a power series on the small parameter $|\mathbf{r}_2 - \mathbf{r}_1|/D$,

$$r_{AB} \simeq D + (\mathbf{n}_D, \mathbf{r}_2 - \mathbf{r}_1) = D + \triangle, \quad \triangle = (\mathbf{n}_D, \mathbf{r}_2 - r_1),\ \mathbf{n}_D = \mathbf{D}/D. \tag{12}$$

Introducing this approximation in (10), considering that the direction $\mathbf{n}_{AB}$ coincides with $\mathbf{n}_D$ we can integrate under the atomic variables $\mathbf{r}_A$, $\mathbf{r}_B$ and the direction of wave vector $\mathbf{k}$. Thus, the exchange integral with the electromagnetic field vacuum depending on lag τ after integration over generalized coordinates of the atoms and the wave vector of photons becomes

$$\begin{aligned} J_{AB}(\tau) = {}& \frac{2^2 e^2 \hbar}{c\pi a^2 \mu^2 D^3} \left\{ \frac{9}{3!} \frac{\partial^3}{\partial (b^2)^3} - \frac{24}{4!} \frac{\partial^4}{\partial (b^2)^4} + \frac{16}{5!} \frac{\partial^5}{\partial (b^2)^5} \right\} \frac{1}{b^2} \\ & \times \Bigg[-D + \frac{c\tau}{2} \ln \left| \frac{c\tau - D}{c\tau + D} \right| - \frac{ic\tau\pi}{2} \theta(D - c\tau) + \frac{D}{4} \left[\exp\left[-\frac{(c\tau + D)b}{a} \right] \right. \\ & \times \mathrm{Ei}\left(\frac{(c\tau + D)b}{a} \right) + \exp\left[\frac{(c\tau + D)b}{a} \right] \mathrm{Ei}\left(-\frac{(c\tau + D)b}{a} \right) \\ & + \exp\left[-\frac{(c\tau - D)b}{a} \right] \mathrm{Ei}\left(\frac{(c\tau - D)b}{a} \right) + \exp\left[\frac{(c\tau - D)b}{a} \right] \mathrm{Ei}\left(-\frac{(c\tau - D)b}{a} \right) \\ & \left. - i\pi \left(\mathrm{sign}(c\tau - D) \exp\left[-\frac{|c\tau - D|b}{a} \right] + \exp\left[-\frac{(c\tau + D)b}{a} \right] \right) \right] \Bigg]_{b=3/2}. \end{aligned} \tag{13}$$

Here $\mathrm{Ei}(x)$ is the exponential integral [17, 18]. From expression (13) it follows that only two terms depend on the classical causality and are proportional to the Heavyside step function $\theta(c\tau - D)$ (the function $\mathrm{sign}(c\tau - D) = 2\theta(c\tau - D) - 1$). The other terms are connected with non-locality of the electromagnetic field in the processes of spontaneous emission. An interesting behavior is observed in the terms proportional with $\ln|c\tau - D|$, $\mathrm{Ei}(\pm(c\tau - D)b/a)$. This logarithmic divergence, which appears at the moment $\tau = D/c$, is reciprocally compensated in the right-hand side of exchange integral (13).

Taking into account (12) and following the integration procedure proposed in paper [7] one can obtain the integral expression for (8)

$$\begin{aligned} J(\tau) = {}& \frac{2^2 e^2 \hbar}{3^6 a^4 \mu^2} \left\{ \frac{\partial}{\partial b} - \frac{3}{2} \frac{\partial^2}{\partial b^2} + \frac{3}{4} \frac{\partial^3}{\partial b^3} \right\} \left[i\pi \exp\left(-\frac{cb\tau}{a} \right) \right. \\ & \left. - \exp\left[-\frac{cb\tau}{a} \right] \mathrm{Ei}\left(\frac{cb\tau}{a} \right) - \exp\left[\frac{cb\tau}{a} \right] \mathrm{Ei}\left(-\frac{cb\tau}{a} \right) \right]_{b=3/2}. \end{aligned} \tag{14}$$

$J(\tau)$ describes the energy exchange of single atom with electromagnetic field vacuum, while $J_{AB}(\tau)$ describes the cooperative exchange of two atoms. The last integral depends on distance D between the atoms. Simple transformations show that

$$K(\tau) = \sum_k \varphi_{12}^j(k) \varphi_{12}^j(-k) \exp(i\omega_k \tau) \frac{g_k^2}{\hbar^2},$$

$$K_{AB}(\tau) = \sum_k \varphi_{12}^A(k) \varphi_{12}^B(-k) \exp(i\omega_k \tau) \frac{g_k^2}{\hbar^2},$$

are reduced to the integrals (8) and (9) respectively: $K(\tau)=-J(\tau)$, $K_{AB}(\tau)=-J_{AB}(\tau)$. One can observe that integrals $J(\tau)$ and $K(\tau)$ do not depend on the atomic position, while $J_{AB}(\tau)$ and $K_{AB}(\tau)$ are function of the distance D between the radiators.

The equation for atomic inversion,

$$\frac{d\langle R_{zA}(t)\rangle}{dt}=-\sum_{j=A,B}\int_0^t d\tau\, J_{j,A}(\tau)\{\langle (R_j^+(t-\tau)-R_j^-(t-\tau))\times(R_A^-(t)+R_A^+(t))\rangle\}+h.c. \tag{15a}$$

and atomic correlators, $\langle O(t)\rangle\equiv\langle R_A^+(t)R_B^-(t)\rangle$, $\langle R_{zA}(t)R_{zB}(t)\rangle$, can be obtained from master equation (7)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\langle (R_A^+(t)+R_A^-(t))(R_B^+(t)+R_B^-(t))\rangle\\&=\sum_{j=A,B}\int_0^t d\tau\{J_{A,j}(\tau)\langle (R_j^+(t-\tau)-R_j^-(t-\tau))R_{zA}(t)(R_B^+(t)+R_B^-(t))\rangle\\&\quad+J_{B,j}(\tau)\langle (R_j^+(t-\tau)-R_j^-(t-\tau))R_{zB}(t)(R_A^+(t)+R_A^-(t))\rangle\}+h.c.,\end{aligned} \tag{15b}$$

$$\begin{aligned}&\frac{d\langle R_{zA}(t)R_{zB}(t)\rangle}{dt}\\&=-\sum_{j=A,B}\int_0^t d\tau\, J_{A,j}(\tau)\{\langle (R_j^+(t-\tau)-R_j^-(t-\tau))(R_A^-(t)+R_A^+(t))R_{zB}(t)\rangle\\&\quad-\sum_{j=A,B}\int_0^t d\tau\, J_{B,j}(\tau)\{(\langle R_j^+(t-\tau)-R_j^-(t-\tau))(R_B^-(t)+R_B^+(t))R_{zA}(t)\rangle\}\\&\quad+h.c.\{A\leftrightarrow B\}.\end{aligned} \tag{15c}$$

Using the properties of these exchange integrals and equations for correlators (15), in the next section the close system of equations that takes into account the retardation in the process of spontaneous emission is proposed.

3 The System of Equations that Describes the Exchange Energy between Two Radiators

In the previous section we had obtained the chain of equations for atomic correlators (15). This chain of equations is obtained using exact integration and the procedure of elimination of boson operators. More than this the equations are obtained without dipole approximation. In the right-hand sides of (15) we must know the dependence of correlators $\langle R_A^+(t)R_B^-(t-\tau)\rangle$, $\langle R_B^+(t)R_A^-(t-\tau)\rangle$, $R_{zA}(t)R_{zB}(t-\tau)$ and $R_{zj}(t-\tau)$ on the retardation time, τ. As we are interested by the initial stage of the emission, it is convenient to neglect the retardation in the slow part of the operators in the right-hand sides of (15). Considering that at the beginning of the emission the inversion can be divided into two parts which corresponds to slowly and rapidly oscillating ones

$$R_{zj}(t-\tau)=\mathfrak{R}_{zj}(t-\tau)+[F_z(t-\tau)\exp[i\omega_0(t-\tau)]+h.c.] \tag{16a}$$

we can neglect the retardation in the expression for $\mathfrak{R}_{zj}(t-\tau)$ and $F_z(t-\tau)$. Taking into account the fact that the amplitude $F_z(t-\tau) \ll \mathfrak{R}_{zj}(t-\tau)$ we consider only the contribution of the first term in (16a). This approximation is true at the initial stage of emission when the slowly oscillating part of the inversion is larger than the rapidly oscillating one. We emphasize that at the initial stage of emission, the inversion for A or B atom is $\langle R_z\rangle = 1/2$. Since the amplitudes of $J(\tau)$, $K(\tau)$ and similar Hermitian conjugate functions, rapidly attenuate with increasing of retardation τ in time intervals that are large compared to D/c, it is convenient to separate the rapidly oscillating part (with respect to τ) in the excitation and annihilation operators $R_j^{\pm}(t-\tau) = \tilde{R}_j^{\pm}(t)\exp[\pm i\omega_0\tau]$. Here the functions $\tilde{R}^{\pm}(t-\tau)$ are assumed to vary smoothly compared to $\exp[i\omega_0(t-\tau)]$ in the sense that $|d\tilde{R}^{\pm}(t-\tau)/d(t-\tau)| \ll |R^{\pm}(t-\tau)|\omega_0$. In this approximation the correlators on the right-hand sides of the chain of (15) can be approximated by the following expressions

$$\begin{aligned}\langle R_j^-(t-\tau)R_j^+(t)\rangle &\simeq \exp(i\omega_0\tau)[1/2-\langle R_{zj}(t)\rangle],\\ \langle R_j^+(t-\tau)R_j^+(t)\rangle &= \exp(-i\omega_0\tau)\langle R_j^+(t)R_j^+(t)\rangle = 0.\end{aligned} \tag{16b}$$

In the same way the correlator $\langle R_A^+(t-\tau)R_B^+(t)\rangle$ and its Hermitian conjugate terms, oscillating with double frequency $\langle R_A^+(t-\tau)R_B^+(t)\rangle \sim \exp(2i\omega_0 t)$, are neglected from the right-hand sides of (15). More than this, the correlator and its derivative haven't the non-zero source. Thus in the second order approximation of the interaction constant we can receive the closed system of equations from (15).

Using the notation $X(t) = \langle R_{zj}(t)\rangle$, $Y(t) = \langle R_A^+(t)R_B^-(t)\rangle$ and $Z(t) = \langle R_{zA}(t)R_{zB}(t)\rangle$ and the approximation (16b) one can obtain the following system of equations from (15)

$$\frac{dX(t)}{dt} = -I(t)[X(t)+1/2] - J(t)[X(t)-1/2] + J_{AB}^s(t)Y(t), \tag{17a}$$

$$\frac{dY(t)}{dt} = -(I(t)+J(t))Y(t) + 2J_{AB}^c(t)Z(t) - J_{AB}^s(t)X(t), \tag{17b}$$

$$\frac{dZ(t)}{dt} = -2(I(t)+J(t))Z(t) - (I(t)-J(t))X(t) + J_{AB}^c(t)Y(t), \tag{17c}$$

where $I(t)$ and $J(t)$ describe the decay rate of individual atoms stimulated by electromagnetic field vacuum

$$I(t) = \int_0^t [J(\tau)\exp(-i\omega_0\tau) + h.c.]d\tau, \qquad J(t) = \int_0^t [J(\tau)\exp(i\omega_0\tau) + h.c.]d\tau. \tag{18a}$$

The collective emission rate coefficients $J_{AB}^c(t)$ and $J_{AB}^s(t)$ stimulated by the exchange processes through the vacuum field are

$$\begin{aligned}J_{AB}^c(t) &= 2\int_0^t [J_{AB}(\tau) + J_{AB}^*(\tau)]\cos(\omega_0\tau)d\tau,\\ J_{AB}^s(t) &= 2i\int_0^t [J_{AB}^*(\tau) - J_{AB}(\tau)]\sin(\omega_0\tau)d\tau.\end{aligned} \tag{18b}$$

This system of equations describes the energy exchange between two atoms in the cooperative spontaneous emission.

Taking into account the expressions for $J(\tau)$ we can integrate the right-hand sides of the equations (18b) over the generalized electron coordinates, the wave vector and the retardation τ. After integration we obtain the following expression for $J^c_{AB}(t)$:

$$\begin{aligned}
J^c_{AB}(t) = &-\frac{2^{12}\hbar e^2}{3^{10}\pi c^2\delta^2\mu^2}k_0^3\hat{E}_{AB}(u)\frac{1}{u^2}\left[6\frac{\sin\omega_0 t}{(k_0D)^2}+3\left(\frac{\omega_0 t\sin\omega_0 t}{(k_0D)^3}+\frac{\cos\omega_0 t}{(k_0D)^3}\right)\ln\left|\frac{t-D/c}{t+D/c}\right|\right.\\
&+3\left(\frac{\cos k_0D}{(k_0D)^3}+\frac{\sin k_0D}{(k_0D)^2}\right)(\mathrm{Ci}[\omega_0(t+D/c)]-\mathrm{Ci}[\omega_0|t-D/c|])\\
&-3\left(\frac{\cos k_0D}{(k_0D)^2}-\frac{\sin(k_0D)}{(k_0D)^3}\right)(\mathrm{Si}[\omega_0(t+D/c)]+\mathrm{Si}[\omega_0|t-D/c|])\\
&-\frac{3\delta}{2(k_0D)^2(u^2+\delta^2)}[(\delta\sin\omega_0 t-u\cos\omega_0 t)\{\exp[-\nu u(t+D/c)]\\
&\times\mathrm{Ei}[\nu u(t+D/c)]+\exp[-\nu u(t-D/c)]\,\mathrm{Ei}[\nu u(t-D/c)]\}\\
&+(\delta\sin\omega_0 t+u\cos\omega_0 t)\{\exp[\nu u(t+D/c)]\,\mathrm{Ei}[-\nu u(t+D/c)]\\
&+\exp[\nu u(t-D/c)]\,\mathrm{Ei}[-\nu u(t-D/c)]\}\\
&+2\delta[(\mathrm{Ci}[\omega_0(t+D/c)]-\mathrm{Ci}[\omega_0|t-D/c|])\sin(k_0D)\\
&\left.-(\mathrm{Si}[\omega_0(t+D/c)]+\mathrm{Si}[\omega_0|t-D/c|])\cos(k_0D)]]\right].
\end{aligned}\tag{19}$$

Similar expression can be obtained for

$$\begin{aligned}
J^s_{AB}(t) = &-\frac{2^{12}\hbar e^2}{3^{10}c^2\delta^2\mu^2}k_0^3\hat{E}_{AB}(u)\frac{1}{u^2}\left[3\theta(t-D/c)\right.\\
&\times\left(\frac{\cos k_0D}{(k_0D)^2}-\frac{\sin(k_0D)}{(k_0D)^3}\right)-3\theta(D/c-t)\left(\frac{\sin\omega_0 t}{(k_0D)^3}-\frac{\omega_0 t\cos\omega_0 t}{(k_0D)^3}\right)\\
&+\frac{3\delta}{2(k_0D)^2(u^2+\delta^2)}[(u\sin\omega_0 t+\delta\cos\omega_0 t)\{\exp[-\nu u(t-D/c)]\theta(t-D/c)\\
&+\exp[-\nu u(t+D/c)]\}+(u\sin\omega_0 t-\delta\cos\omega_0 t)\\
&\left.\times\exp[-\nu u(D/c-t)]\theta(D/c-t)-2\delta\cos k_0D]\right].
\end{aligned}\tag{20}$$

In a similar way we can find explicit values for $I(t)$ and $J(t)$ from (18a)

$$I(t)=-\frac{2^7\alpha^3 c}{3^8 a}\hat{E}(u)(L(u,t)+M(u,t))|_{u=1},\qquad J(t)=-\frac{2^7\alpha^3}{3^8 a}\hat{E}(u)M(u,t)|_{u=1},\tag{21}$$

where

$$L(u,t)=\frac{1}{(u^2+\delta^2)}[\delta-\exp(-\nu ut)(u\sin\omega_0 t+\delta\cos\omega_0 t)],\tag{22}$$

$$\begin{aligned} M(u,t) = \frac{1}{(u^2+\delta^2)} \Bigg[& \bigg[\frac{u}{\pi}(\exp(-\nu ut)\,\mathrm{Ei}(\nu ut) - \exp(\nu ut)\,\mathrm{Ei}(-\nu ut)) \\ & + \delta\exp(-\nu ut)\bigg]\cos\omega_0 t + \bigg[u\exp(-\nu ut) - \frac{\delta}{\pi}(\exp(-\nu ut)\,\mathrm{Ei}(\nu ut) \\ & + \exp(\nu ut)\,\mathrm{Ei}(-\nu ut))\bigg]\sin\omega_0 t + \delta\frac{2}{\pi}\left(\mathrm{Si}\,\omega_0 t - \frac{\pi}{2}\right)\Bigg], \end{aligned} \tag{23}$$

$\nu = 3c/2a$, $\delta = 2\omega_0 a/3c$, $u = 1$, $\alpha = e^2/\hbar c$, α is the hyperfine structure constant, $\mathrm{Si}(x)$ and $\mathrm{Ci}(x)$ are the sine and cosine integrals [17], and the operators $\hat{E}_{AB}(u)$, $\hat{E}(u)$ are determined by expressions

$$\hat{E}_{AB}(u) = \frac{\partial^3}{\partial(u^2)^3} + \left(\frac{2}{3}\right)^3 \frac{\partial^4}{\partial(u^2)^4} + \frac{1}{5}\left(\frac{2}{3}\right)^6 \frac{\partial^5}{\partial(u^2)^5}, \tag{24}$$

and

$$\hat{E}(u) = \frac{\partial}{\partial u} - \frac{\partial^2}{\partial u^2} + \frac{1}{3}\frac{\partial^3}{\partial u^3}. \tag{25}$$

As follows from these numerical results represented in paper [7], the coefficients which describe the interaction of individual atoms with electromagnetic field vacuum $I(t)$ and $J(t)$ have a damping oscillatory behavior. The damping time with magnitude $\tau \sim 1/\omega_0$ implies that the emerging wave packet (photon) has a strong effect not only on the excited state of the atom $\langle R_z(t)\rangle + 1/2$, but on the ground state too $-\langle R_z(t)\rangle + 1/2$. It can easily be shown that the coefficient of the stimulating action of the emerging photon field, $J(t)$, is of the same order of magnitude as the spontaneous emission Einstein coefficient $A_{II} = 1/\tau_0$ only for $a/c \sim t \lesssim 1/\omega_0$. Since the atom is several thousand times smaller than the radiation wavelength, $a \ll \lambda_0 = 2\pi c/\omega_0$, the expressions for $I(t)$ and $J(t)$ simplify when $t \gg a/c$. Expanding the function $1/(u^2+\delta^2)$ in a power series in the small parameter δ

$$\frac{1}{u^2+\delta^2} = \frac{1}{u^2}\left(1 - \left(\frac{\delta}{u}\right)^2 + \left(\frac{\delta}{u}\right)^4 - \cdots\right) \tag{26}$$

and allowing for the asymptotic behavior of the exponential integral at large value of the argument ($t > 2a/3c$) [7, 22]

$$\mathrm{Ei}(\nu ut) = \frac{e^{\nu ut}}{\nu ut}\left(1 + \frac{1}{\nu ut} + \frac{2!}{(\nu ut)^2} + \cdots\right) \tag{27}$$

we arrive at the following analytical expressions for the above coefficients to the lowest order in the smallness parameter $(\nu ut)^{-1}$

$$I(t) = \frac{1}{\tau_0} + J(t), \qquad J(t) = B(t) = \frac{1}{\pi\tau_0}\left(\mathrm{Si}(\omega_0 t) - \pi/2 + \frac{\cos\omega_0 t}{\omega_0 t}\right), \tag{28}$$

where

$$\tau_0 = 3^{11}\hbar c^3/(2^{17}e^2a^2\omega_0^3) \tag{29}$$

is the spontaneous emission rate of the atom. The damping oscillatory behavior of (21) on time interval $t > 2a/3c$ is approximately described by the function $\mathrm{Si}(\omega_0 t) + \cos(\omega_0 t)/\omega_0 t$.

Let us estimate the cooperative terms described in expressions (19) and (20). Taking into account the terms proportional with $1/u^2$ in the right-hand sides of these expressions in the same approximation we obtain the following exchange rate coefficients between the radiators A and B

$$\begin{aligned} J^c_{AB}(t) = -\frac{1}{\tilde{\tau}_0 \pi}\Bigg[& 6\frac{\sin\omega_0 t}{(k_0 D)^2} + 3\left(\frac{\omega_0 t\sin\omega_0 t}{(k_0 D)^3} + \frac{\cos\omega_0 t}{(k_0 D)^3}\right)\ln\left|\frac{t - D/c}{t + D/c}\right| \\ & + 3\left(\frac{\cos k_0 D}{(k_0 D)^3} + \frac{\sin k_0 D}{(k_0 D)^2}\right)(\mathrm{Ci}[\omega_0(t + D/c)] - \mathrm{Ci}[\omega_0|t - D/c|]) \\ & - 3\left(\frac{\cos k_0 D}{(k_0 D)^2} - \frac{\sin(k_0 D)}{(k_0 D)^3}\right)(\mathrm{Si}[\omega_0(t + D/c)] + \mathrm{Si}[\omega_0|t - D/c|])\Bigg], \end{aligned} \tag{30}$$

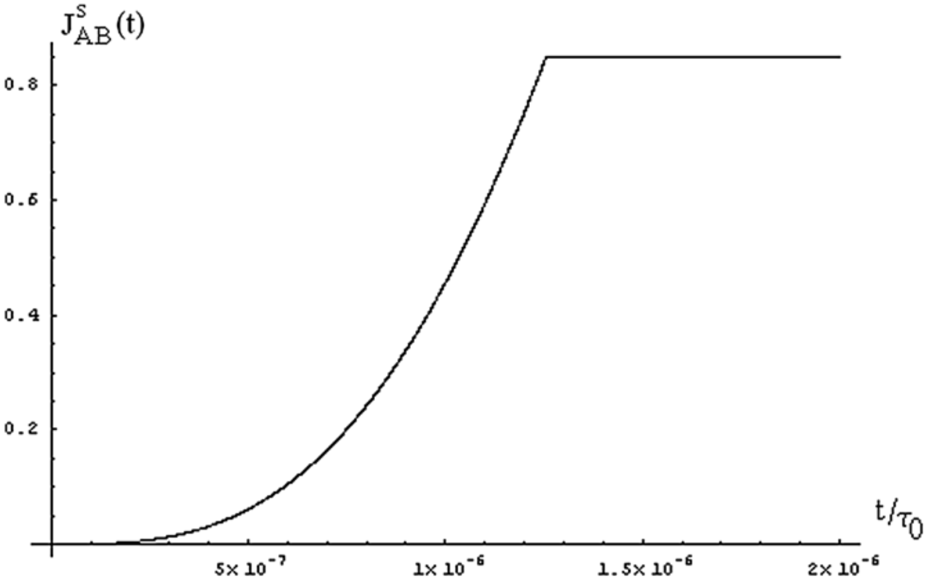
$$\begin{aligned} J^s_{AB}(t) = -\frac{1}{\tilde{\tau}_0}\Bigg[& 3\theta(t - D/c)\left(\frac{\cos k_0 D}{(k_0 D)^2} - \frac{\sin(k_0 D)}{(k_0 D)^3}\right) \\ & - 3\theta(D/c - t)\left(\frac{\sin\omega_0 t}{(k_0 D)^3} - \frac{\omega_0 t\cos\omega_0 t}{(k_0 D)^3}\right)\Bigg]. \end{aligned} \tag{31}$$

Here $\tilde{\tau}_0 = 81\tau_0/121$ a little bit differs from τ_0. This divergence is connected with new approach in the calculation of matrix elements proposed in (11–13). The time behavior of exchange integrals $J^s_{AB}(t)$ and $J^c_{AB}(t)$, is plotted in Figs. 2a and b respectively for the small value of the D distance between the atoms in comparison with the λ wave length: $D/\lambda = 0.2$, $\omega_0\tau_0 = 10^6$. As follows from Fig. 2a, the coefficient $J^s_{AB}(t)$ tends to a stable amplitude. The coefficient $J^c_{AB}(t)$ has an oscillatory behavior with decreasing amplitude due to the fact that with increasing the time, the function $\ln|[t - D/c]/[t + D/c|$ tends to zero value.

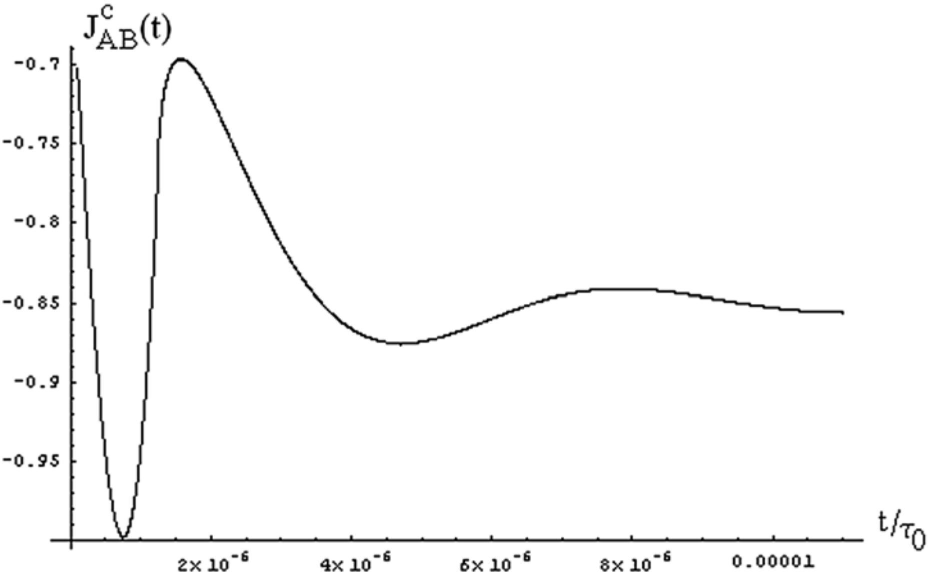
Taking into account analytical expressions (30) and (31) one can numerically study the behavior of two atoms situated at a larger distance in comparison with the wave length ($D/\lambda = 4$, $\omega_0\tau_0 = 10^6$). From numerical studies of the time-dependent solutions of the coefficients $J^s_{AB}(t)$ and $J^c_{AB}(t)$, it follows that for time intervals $t < D/c$ the interaction between atoms A and B exists due to non-locality of electromagnetic field (see Figs. 3a and b). From these numerical results we can observe that exchange integrals have an oscillatory behavior for $t < D/c$. With increasing the time, $t > D/c$, the value of exchange integral $J^s_{AB}(t)$ becomes stabilized, while $J^c_{AB}(t)$ continue its oscillatory behavior with decreasing amplitude. The numerical solutions for the system of (18) are plotted in Figs. 4a, b for the same parameters.

From Fig. 4b it follows that the correlator $\langle R^+_A(t)R^-_B(t)\rangle$ oscillates with the damped amplitude plotted in Fig. 3. The time behavior of the correlator $\langle R_{zA}(t)R_{zB}(t)\rangle$, plotted in Fig. 4c, describes the population correlations between atoms A and B. As follows from Fig. 4c, the entanglement between atomic inversions is established at the early stage of the decay process.

From analytical and numerical results it follows that the exchange energies between the radiators at an earlier stage of evolution becomes very important. We must take into account the time average for square values of these coefficients in the second equation of the system (18). This is possible if we develop a theory in which the variables X, Y and Z contain a smooth part and an oscillatory part. Neglecting this oscillatory part we return to the traditional Born-Marcoff approximation. Indeed, averaging over the time the coefficients in the system of (18) $\bar{J}^c_{AB}(t) \simeq \bar{J}^s_{AB}(t) = 1/\tilde{\tau}_1$; $\bar{I}(t) = 1/\tau_0$, one can obtain the following system



(a)

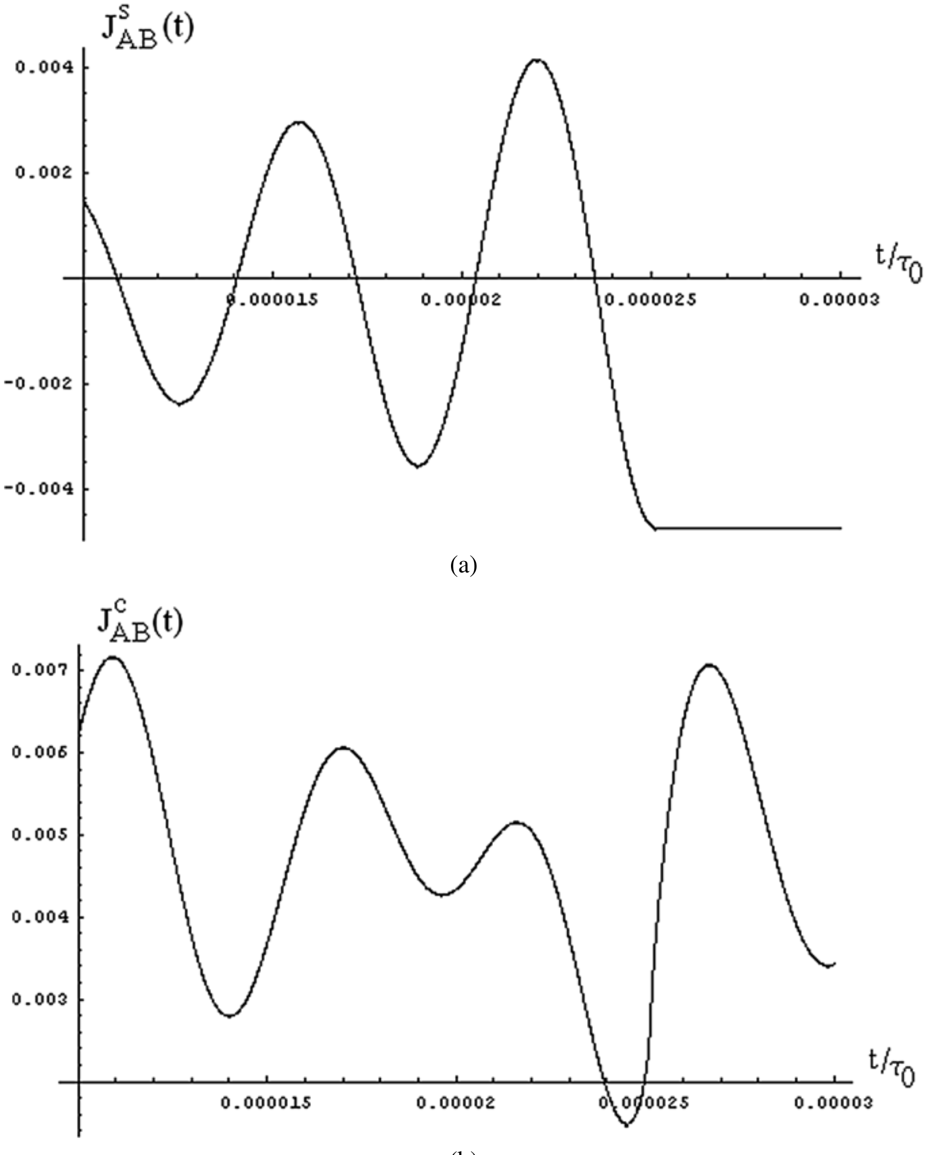


(b)

Fig. 2 The time behavior of exchange integrals $J_{AB}^{s}(t)$ and $J_{AB}^{c}(t)$ for the following values of parameters: $D/\lambda = 0.2$, $\omega_0\tau_0 = 10^6$. The coefficient $J_{AB}^{s}(t)$ tends to a stable amplitude. The coefficient $J_{AB}^{c}(t)$ has an oscillatory behavior with decreasing amplitude due to the fact that with increasing the time, the function $\ln|[t - D/c]/[t + D/c]|$ tends to zero value

of equations for two-level atoms placed at the distance D in cooperative interaction

$$\frac{d}{dt}X(t) = -\frac{1}{\tau_0}\left(X(t) + \frac{1}{2}\right) - \frac{1}{\tilde{\tau}_1}Y(t), \tag{32a}$$



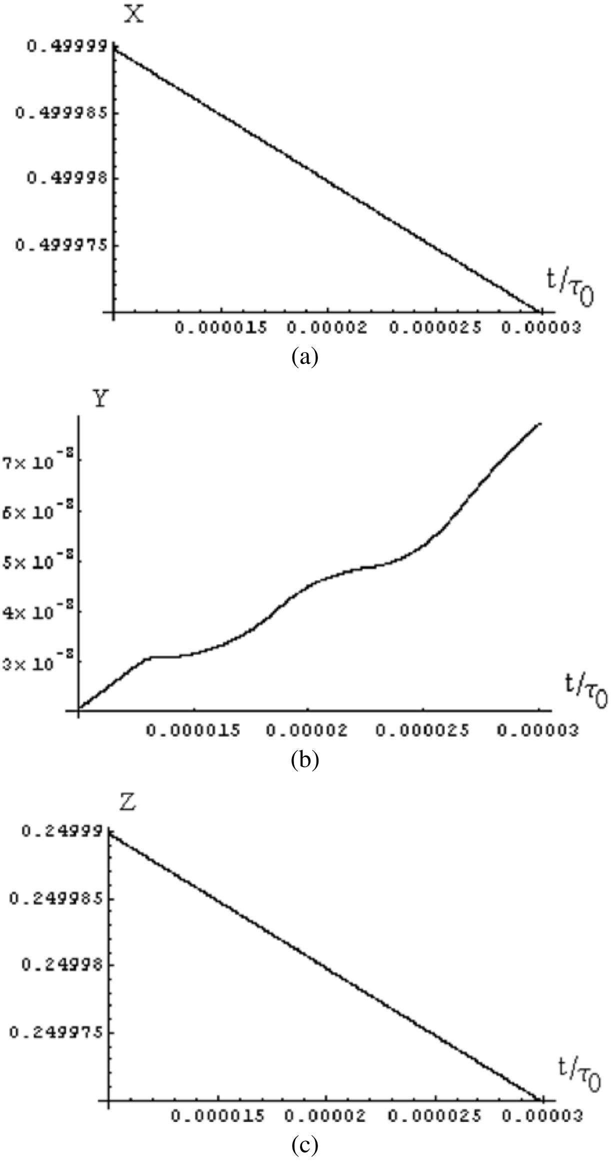
(a)

(b)

Fig. 3 The time behavior of exchange integrals $J^s_{AB}(t)$ and $J^c_{AB}(t)$ for the following values of parameters: $D/\lambda = 4.0$, $\omega_0\tau_0 = 10^6$. With increasing the time $t > D/c$, the value of exchange integral $J^s_{AB}(t)$ becomes stabilized, while the exchange integral $J^c_{AB}(t)$ continue its oscillatory behavior with decreasing amplitude

$$\frac{d}{dt}Y(t) = -\frac{1}{\tau_0}Y(t) + \frac{1}{\tilde{\tau}_1}X(t) + \frac{2}{\tilde{\tau}_1}Z(t), \tag{32b}$$

$$\frac{d}{dt}Z(t) = -\frac{2}{\tau_0}Z(t) - \frac{1}{\tau_0}X(t) + \frac{1}{\tilde{\tau}_1}Y(t). \tag{32c}$$



(a)

(b)

(c)

Fig. 4 The numerical solutions for the system of (18) for the same parameters. The correlator $\langle R_A^+(t)R_B^+(t)\rangle$ oscillates with the amplitude, which depends on the oscillatory behavior of exchange integrals $J_{AB}^s(t)$ and $J_{AB}^c(t)$

Here we introduce the new variables: $X(t) = (\langle R_{z1}(t)\rangle + \langle R_{z2}(t)\rangle)/2$ is the inversion of atoms, $Y(t) = (\langle R_2^+(t)R_1^-(t)\rangle + \langle R_1^+(t)R_2^-(t)\rangle)/2$ is the cooperative rate of emission, $Z(t) = \langle R_{z1}(t)R_{z2}(t)\rangle$ is the correlation function of inversions for the first atom and the second one.

$$\frac{1}{\tilde{\tau}_1} = \frac{3}{\tau_0}\left[\frac{\cos k_0 D}{(k_0 D)^2} - \frac{\sin(k_0 D)}{(k_0 D)^3}\right]$$

is the cooperative decay rate.

Using the initial conditions for variables: $X(t=0)=1/2$, $Y(t=0)=0$, $Z(t=0)=1/4$ the following solution of the system (32) is obtained:

$$\begin{aligned} Z(\tau) &= -\frac{4J^2}{1-J^2}\exp(-2\tau)+\frac{1-J}{1+J}\exp[-(1-J)\tau] \\ &\quad +\frac{1+J}{1-J}\exp[-(1+J)\tau]-1/2, \end{aligned} \tag{33a}$$

$$\begin{aligned} V(\tau) &= -\frac{4J}{1-J^2}\exp(-2\tau)-\frac{1-J}{1+J}\exp[-(1-J)\tau] \\ &\quad +\frac{1+J}{1-J}\exp[-(1+J)\tau], \end{aligned} \tag{33b}$$

$$\begin{aligned} Y(\tau) &= \frac{1+J^2}{1-J^2}\exp(-2\tau)-\frac{1-J}{2(1+J)}\exp[-(1-J)\tau] \\ &\quad -\frac{1+J}{2(1-J)}\exp[-(1+J)\tau]+1/4, \end{aligned} \tag{33c}$$

where $J=\tau_0/\tau_1$, $\tau=t/\tau_0$.

Let us compare the dependence of the cooperative exchange integral between the atoms at distance D with expressions obtained in literature [10]

$$\frac{1}{\tilde{\tau}_1}=\frac{3}{2\tau_0}\left[[1-(\mathbf{n}_d\cdot\mathbf{n}_D)^2]\frac{\sin(k_0D)}{k_0D}+[1-3(\mathbf{n}_d\cdot\mathbf{n}_D)^2]\left[\frac{\cos(k_0D)}{(k_0D)^2}-\frac{\sin(k_0D)}{(k_0D)^3}\right]\right], \tag{34}$$

where $\mathbf{n}_d$ and $\mathbf{n}_D$ are the unitary vectors. Averaging this expression in the direction of dipole ε_a we can obtain the following expression:

$$\frac{1}{\tau}=\frac{1}{\tau_0}\frac{\sin k_0D}{k_0D}.$$

In the case when the atoms are located within a volume with linear dimensions smaller compared with radiation wavelength the result for τ_1 and $\tilde{\tau}_1$ tends to τ_0 and $\tilde{\tau}_0$.

4 Conclusions

From the proposed approach it follows that, in cooperative spontaneous emission, retardation plays an important role and cannot be neglected in the processes of emission of two radiators. For time $t\sim D/c$ the non-Marcovian approach in the study of cooperative spontaneous emission is proposed. The method takes into account the simultaneous interaction of two atoms formed from nuclei and electrons in interaction with electromagnetic field. The averaging over the emission direction and dipole approximation was made at a later stage of the system of kinetic equations (15). This method gives us the possibility to distinguish causal and non-causal exchange processes between the entangled atoms influenced by electromagnetic field vacuum.

A new method of elimination of electromagnetic field operators was proposed. The method is not traditionally: firstly from the master equation (equation for atomic variables) are eliminated electromagnetic field variables and after that are integrated over the atomic variables taking into account the same direction of quantification for both atoms, coinciding with the axis between the radiators. This procedure of integration gives us another result for exchange integral even for long time intervals in comparison with traditionally proposed in the literature. The system of equations that describes the behavior of spontaneous emission of two atoms for time intervals, that are larger than retardation times was obtained.

Acknowledgements This work was supported by founds provided by National Grant No. 06.408.013F and Young Scientists Grant No. 07.408.31 of Academy of Sciences of Moldova.

References

1. Allen, L.C., Eberly, J.H.: Optical resonance and two level atoms. Science (1987)
2. Atkins, P.W., Wooloy, R.G.: Proc. R. Soc. Lond. A **319**, 549 (1970)
3. Berkeland, D.J., Raymondson, D.A., Tassin, V.M.: Phys. Rev. A **69**, 052103 (2004)
4. Beugnon, J., Jones, M.P.A., Dingjan, J., Darquie, B., Messin, G., Browaeys, A., Grangier, P.: Nature (Lond.) **440**, 779–782 (2006)
5. Davidovich, L., Nussenzeig, H.M.: In: Barut, A.O. (ed.) Foundation of Radiation Theory and Quantum Electrodynamics. Plenum, New York (1980)
6. DeVoe, R.G., Brewer, R.G.: Phys. Rev. Lett. **76**, 2049 (1996)
7. Enaki, N.A.: JETP **82**, 607–615 (1996)
8. Facchi, P., Pascazio, S.: Physica A **271**, 271 (1999)
9. Ficek, Z., Tanas, R.J.: J. Mod. Opt. **50**, 2765 (2003)
10. Gross, M., Haroche, S.: Phys. Rep. **93**, 302 (1982)
11. Khalfin, L.A.: Dokl. Akad. Nauk SSSR **111**, 345 (1956)
12. Khalfin, L.A.: Dokl. Akad. Nauk SSSR **115**, 2770 (1957)
13. Khalfin, L.A.: Zh. Eksp. Teor. Fiz. **33**, 1351 (1957)
14. Loudon, R.: The Quantum Theory of Light. Clarendon Press, Oxford (1973)
15. Millonni, P.W., Knight, P.L.: Phys. Lett. A **56**, 275 (1976)
16. Milonni, P.W.: Am. J. Phys. **52**, 340 (1984)
17. Prudnikov, A.P., Brychkov, Y.A., Marichev, O.I.: Handbook of Mathematical Functions. Nauka, Moscow (1979). Edited by M. Abramowitz, I. Stigan (in Russian)
18. Prudnikov, A.P., Brychkov, Y.A., Marichev, O.I.: Integrals and Series. Nauka, Moscow (1981). (In Russian)
19. Seke, J., Herfort, W.N.: Phys. Rev. A **38**, 833 (1989)
20. Seke, J., Herfort, W.N.: Phys. Rev. A **40**, 1926 (1988)
21. Wilkievich, K., Eberly, J.H.: Ann. Phys. **101**, 574 (1976)
22. Wooloy, R.G.: Proc. R. Soc. Lond. A **321**, 557 (1971)
23. Zhao, H., Ujihara, K.: J. Mod. Opt. **53**, 835 (2006)